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FEATURES OF VISCOUS-FLUID FLOW IN AN ELASTIC PIPELINE

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Based on N. E. Zhukovskii's investigations of fluid flow in a pipe whose walls can expand, the basic hydrodynamic equations in such a pipeline have been found with the vector form of the Newton law for a viscous fluid. The analog of the Poiseuille formula for the fluid flow rate in an elastic pipeline has been determined. A comparative analysis of the viscous flow in elastic and rigid pipelines has been made. It has been shown that the relationship between the maximum velocity of the fluid in the pipeline cross section and the cross-section-average velocity is the same for elastic and rigid pipelines.

Keywords: *elastic pipeline, Zhukovskii momentum equation, viscous fluid, law of friction in the flow, fluid flow rate.*

Introduction. Investigation of flow of a viscous fluid in an elastic pipeline is a much more difficult problem than studying an analogous flow of an ideal fluid. Certain of the primary aspects of such investigation have been given in [1].

The present work seeks to analyze the influence of viscosity on the basic hydrodynamic equations in an elastic pipeline. Flow in an elastic pipeline was analyzed by N. E. Zhukovskii [2] with the integral of the momentum equation for water in a pipe whose walls can expand over the longitudinal coordinate x. If, as has been done in [3], the momentum equation proposed by N. E. Zhukovskii is written in differential form, the term with fluid pressure P must be in proportion to $\partial(PS_x)/\partial x$, not to $\partial P/\partial x$. Within the framework of this idea, it is not necessary to consider the stress tensor for description of viscous flow in the elastic pipeline, since the diagonal tensor components are equal to the pressure rather than to the projections of the driving force.

Momentum Equation in the Elastic Pipeline. We write the momentum equation for the fluid flow in the elastic pipeline in vector form using the second Newton law as

$$\mathbf{a}dm = -\,d\mathbf{F}_{\rm d} - d\mathbf{F}_{\rm fr}\,. \tag{1}$$

The friction force $d\mathbf{F}_{\rm fr}$ (Fig. 1) is opposite to the motion (acceleration) of the fluid; therefore, it has been taken with a minus sign. The quantity determining the driving force is $-d\mathbf{F}_{\rm d} > 0$. This is due to the fact that $\Delta\mathbf{F}_{\rm d} = \mathbf{F}_2 - \mathbf{F}_1 < 0$, since the force $\mathbf{F}_2$ is less than $\mathbf{F}_1$ in motion of the fluid in the positive direction of the x axis.

Taking $dm = \rho dW$ into account and writing the total derivative as the sum of the local and convective components [4]

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\,\mathbf{V}\,, \tag{2}$$

we can represent Eq. (1) as

$$\frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\,\mathbf{V} = -\frac{1}{\rho}\frac{d\mathbf{F}_{\rm d}}{dW} - \frac{1}{\rho}\frac{d\mathbf{F}_{\rm fr}}{dW}\,. \tag{3}$$

Equation (3) has been written for the arbitrary direction of the fluid velocity $\mathbf{V}$. We consider first the term determined by the friction force $d\mathbf{F}_{\rm fr}$, i.e., by the fluid viscosity, on the right-hand side of (3).

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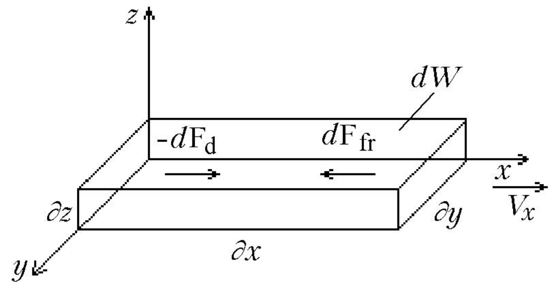


Fig. 1. Forces acting on the separated fluid volume dW in the flow.

Newton Law for the Viscous Fluid. Traditionally the momentum equation for a viscous fluid is initially written in tensor form [5]. Then a stress tensor that is related to the strain-rate tensor is introduced [6]. Such logic of analysis is common for both the strain of a rigid body [7] and viscous-fluid flow. The only difference is that for the rigid body, the relationship of the tensors is determined by Hooke's law, whereas for the fluid, it is determined by Newton's law.

However in such an approach for the fluid, there is an ambiguity in the sign of the law relating the components of tangential stresses in the stress tensor and the corresponding components of the strain-rate tensor. The relationship of the stress and strain-rate tensors in a form suitable for use in an arbitrary coordinate system, including the case of rotation of the coordinate axes, must generally be represented with a plus sign [5, 6]. At the same time, in solving specific problems, e.g., Hagen–Poiseuille flow in a pipe, it is convenient to use the minus sign in Newton's law relating the tangential friction stresses and the derivative of velocity with respect to the coordinate for the viscous fluid [6, 9]:

$$\tau = -\eta \frac{\partial V_x}{\partial z} . \tag{4}$$

We note that with allowance for the sign of (4), the component of the strain-rate tensor in [6, 9] can be written in the form $\tau_{zx} = \tau_{xz} = -\eta\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)$. In [8, 10], the sign ambiguity is compensated for by the modulus of the derivative of the fluid velocity with respect to the coordinate. Physically the ambiguity in the sign, e.g., in calculating pipe flow, is due to the fact that tangential friction stresses in the Newton law for a viscous fluid are directed streamwise, if we consider the action of a fluid layer that is closer to the center of the flow on a layer that is at a distance from the center. If we bear in mind the action of the more peripheral layer on the layer that is closer to the center, the friction force will be opposite to the fluid flow.

In connection with the above remarks, in transformations of (3), we will not start from the tensor form of the Newton law for the viscous fluid and will make the entire analysis in vector form without introducing the tangential friction stresses. We write the vector form of the Newton law for the viscous fluid as

$$d\mathbf{F}_{\text{fr}} = \eta d\mathbf{S} \times \text{rot}\,\mathbf{V} . \tag{5}$$

For the sake of convenience we perform all preliminary transformations in a Cartesian coordinate system. We note that in potential flow, i.e., when rot $\mathbf{V}$ = 0, the friction force is $d\mathbf{F}_{\text{fr}}$ = 0, i.e., the flow is isentropic [5]. Friction between the fluid layers is absent not only in the case of an ideal fluid but also in the case of equal velocity of these layers.

Figure 2 shows the right-hand set of three vectors in formula (5). Using the rules of vector algebra [4], we rewrite (5) in projections

$$dF_x\mathbf{i} + dF_y\mathbf{j} + dF_z\mathbf{k} = \eta \left\{ \left[dS_x \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) - dS_z \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) \right] \mathbf{i} \right.$$

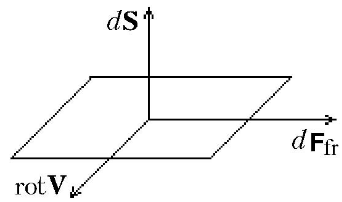


Fig. 2. Right-hand set of three vectors in the vector product of Newton's law for the viscous fluid.

$$+\left[dS_z\left(\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right)-dS_x\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)\right]\mathbf{j}+\left[dS_x\left(\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial y}\right)-dS_y\left(\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right)\right]\mathbf{k}\Bigg\}. \quad (6)$$

Formula (6) yields

$$dF_x=\eta\left[dS_y\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)-dS_z\left(\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right)\right]. \quad (7)$$

For fluid flow between two plane walls, we can set $dS_y = 0$ and $\partial V_z/\partial x = 0$ in the coordinates (see Fig. 1). Consequently, in this case the Newton law for the viscous fluid will be written as

$$dF_x=-\eta dS_z\frac{\partial V_x}{\partial z}. \quad (8)$$

We integrate Eq. (1) over the entire volume W of the flowing fluid

$$\rho\int_W \mathbf{a}dW=-\int_W d\mathbf{F}_\mathrm{d}-\int_W d\mathbf{F}_\mathrm{fr}. \quad (9)$$

Let us assume that the driving force is absent, i.e., the flow is inertial in character, $-dF_\mathrm{d} = 0$. In this case we have

$$\rho\int_W \mathbf{a}dW=-\int_W d\mathbf{F}_\mathrm{fr}=-\eta\int_\mathbf{S} d\mathbf{S}\times\mathrm{rot}\,\mathbf{V}. \quad (10)$$

Applying the rotor theorem [4] to the right-hand side of (10), we obtain

$$\rho\int_W \mathbf{a}dW=-\eta\int_\mathbf{S} d\mathbf{S}\times\mathrm{rot}\,\mathbf{V}=-\eta\int_W \mathrm{rotrot}\,\mathbf{V}dW. \quad (11)$$

In the volume integrals in (11), we can equate the integrands

$$\mathbf{a}=-\nu\,\mathrm{rotrot}\,\mathbf{V}=-\nu\,(\mathrm{graddiv}\,\mathbf{V}-\Delta\mathbf{V}). \quad (12)$$

We have used the well-known formula of vector analysis [4]; $\Delta\mathbf{V}$ is the velocity-vector Laplacian. In the case of an incompressible fluid we have div $\mathbf{V} = 0$ [5]. Thus, with account for (3) and on condition that $-d\mathbf{F}_\mathrm{d} = 0$, we have

$$\mathbf{a}=-\frac{1}{\rho}\frac{d\mathbf{F}_\mathrm{fr}}{dW}=\nu\Delta\mathbf{V}. \quad (13)$$

The vector form of the momentum equation, which holds true for flow in both the rigid pipeline and the elastic pipeline, appears as

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\,\mathbf{V} = -\frac{1}{\rho}\frac{d\mathbf{F}_\mathrm{d}}{dW} + \nu\Delta\mathbf{V}\,. \tag{14}$$

Poiseuille Formula for the Elastic Pipeline. In [1], it has been shown that the term $-\frac{1}{\rho}\frac{d\mathbf{F}_\mathrm{d}}{dW}$, which determines the driving force of the fluid flow in the elastic pipeline, must be written, in projection onto the x axis, as

$$-\frac{1}{\rho}\frac{dF_\mathrm{d}}{dW} = -\frac{1}{\rho}\frac{\partial\,(PS_x)}{S_x\partial x}\,. \tag{15}$$

Representation of the term with pressure in the form (15) holds true under a relatively low strain of the walls of the elastic pipeline where the contribution of the forces of reaction of the walls to the fluid in the direction of the x axis is relatively small on the portion of length dx. When the drain of the pipeline is considerable, the stressed state stain in the fluid becomes more complex and can be expressed only by tensor quantities. With account for (15), Eq. (14) in projection onto the x axis has the form

$$\frac{dV_x}{dt} + \frac{1}{\rho}\frac{\partial\,(PS_x)}{S_x\partial x} = \nu\Delta V_x\,. \tag{16}$$

For pipeline flow, it is convenient to use the hydrodynamic equations in the approximation of a boundary layer in cylindrical geometry

$$\frac{dV_x}{dt} + V_x\frac{dV_x}{dx} + V_r\frac{dV_x}{dr} + \frac{1}{\rho}\frac{\partial\,(PS_x)}{S_x\partial x} = \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{dV_x}{dr}\right), \tag{17}$$

$$\frac{\partial\,(rV_x)}{\partial x} + \frac{\partial\,(rV_r)}{\partial r} = 0\,, \tag{18}$$

where (18) is the continuity equation for the incompressible fluid [6].

Let us consider the manner in which we can transform the term with pressure in Eq. (17), eliminating the pressure. We can do this on the horizontal plane plate in the boundary layer, finding the derivative of the Bernoulli equation $P + \frac{\rho U^2}{2}$ = const [6]:

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = U\frac{\partial U}{\partial x}\,. \tag{19}$$

The projection of Eq. (1) onto the x axis for an ideal fluid in the elastic pipeline appears as follows:

$$\partial\,(PS_x) = -\frac{dV_x}{dt}\rho S_x\partial x\,. \tag{20}$$

Denoting the velocity of the pressure wave as $c = \partial x/\partial t$, we find

$$\partial\,(PS_x) = -\rho c S_x dV_x = -\rho c\,(\partial Q - V_x\partial S_x)\,, \tag{21}$$

where the increment in the fluid flow rate in the pipeline is $\partial Q = \partial(V_x S_x) = S_x\partial_x V_x + V_x\partial S_x$. Using the relation $c = -\partial Q/\partial S_x$ [1], we transform formula (21) to the form

$$\partial\,(PS_x) = \rho c^2\left(1 + \frac{V_x}{c}\right)\partial S_x\,. \tag{22}$$

Formula (22) for the viscous fluid in the elastic pipeline should be written as

$$\partial\,(PS_x) = \rho c^2 \left(1 + \frac{\overline{V}_x}{c}\right)\partial S_x = D\left(1 + \frac{\overline{V}_x}{c}\right)\partial S_x\,, \tag{23}$$

where it has been taken into account that the coefficient of elasticity of the pipeline walls is $D = \rho c^2$ [11]. Substituting (23) into (16), we find the momentum equation for the flow in the elastic pipeline

$$\frac{dV_x}{dt} + \frac{D}{\rho}\left(1 + \frac{\overline{V}_x}{c}\right)\frac{\partial S_x}{S_x \partial x} = \nu\,\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right). \tag{24}$$

Using Hooke's law for the elastic pipeline in the form $\partial P = D\dfrac{\partial S_x}{S_x}$ [11], we obtain

$$\frac{dV_x}{dt} + \frac{1}{\rho}\left(1 + \frac{\overline{V}_x}{c}\right)\frac{\partial P}{\partial x} = \nu\,\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right). \tag{25}$$

In this case, we should adopt the plus sign for the relative strain of the cross-sectional area of the pipeline $\partial S_x/S_x$ in Hooke's law. The reason is that the cross-sectional area of the elastic pipeline increases with fluid pressure P in it. A minus sign in Hooke's law is taken in the case where the problem of occurrence of a vibrational or wave process on the walls of the elastic pipeline is analyzed. Here the quantity P represents the reaction of an elastic wall to internal pressure, i.e., is an analog of the restoring force in vibration theory [11].

Equation (25) directly contains the pressure gradient. Therefore, it is convenient for analysis of fluid flows in the elastic pipeline when the pressure gradient and the flow rate are measured. Equation (25) becomes an equation for a rigid pipe when $c >> \overline{V}_x$, i.e., when the velocity of the pressure wave is very large compared to the velocity of the fluid flow. This condition is observed in the rigid pipeline, when the coefficient of elasticity of its walls (and hence their Young's modulus) is $D = \rho c^2 \to \infty$.

In closing, we find the rate of flow of the fluid through the elastic pipeline, i.e., the analog of the Poiseuille formula that is used for rigid pipelines. In the stationary case Eq. (25) appears as

$$\frac{1}{\rho}\left(1 + \frac{\overline{V}_x}{c}\right)\frac{\partial P}{\partial x} = \nu\,\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right). \tag{26}$$

In accordance with [5], we integrate Eq. (26) twice for a constant pressure difference on the portion Δl of the pipeline

$$V_x = \frac{r^2}{4\eta}\left(1 + \frac{\overline{V}_x}{c}\right)\frac{\Delta P}{\Delta l} + a \ln r + b\,. \tag{27}$$

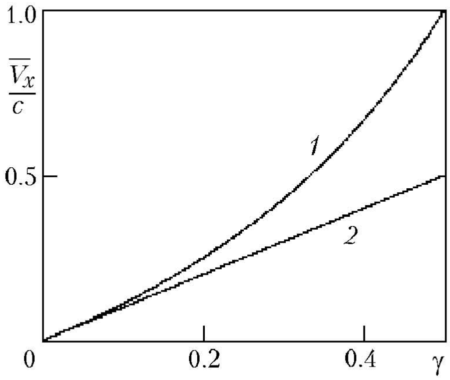
The constant a is equal to zero since the fluid velocity on the pipe axis is finite. The constant b will be found from the condition $V_x = 0$ at $r = R$. Consequently, we have

$$V_x = -\frac{R^2 - r^2}{4\eta}\left(1 + \frac{\overline{V}_x}{c}\right)\frac{\Delta P}{\Delta l}\,, \tag{28}$$

where the minus sign reflects the fact that $V_x > 0$ at $\Delta P/\Delta l < 0$. The fluid flow rate will be found from the formula

$$Q = 2\pi\int_0^R V_x r dr = -\frac{\pi R^4}{8\eta}\left(1 + \frac{\overline{V}_x}{c}\right)\frac{\Delta P}{\Delta l}\,. \tag{29}$$

Setting $Q = \overline{V}_x S_x = \overline{V}_x \pi R^2$, we find, from Eq. (29), the flow velocity average over the pipeline cross section



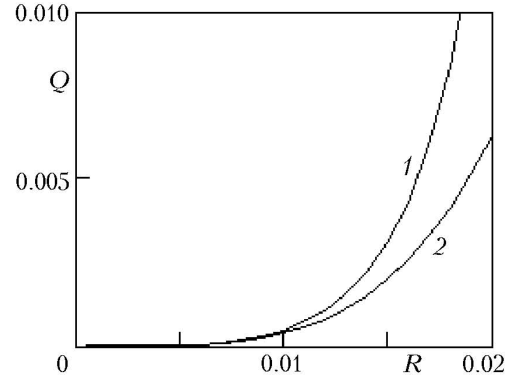


Fig. 3. Ratio of the average velocity of the viscous fluid to the velocity of the pressure wave c vs. dimensionless number γ in the pipes: 1) elastic pipe; 2) rigid pipe.

Fig. 4. Rate of flow of the fluid in the pipes vs. pipeline radius: 1) elastic pipe; 2) rigid pipe. Q, m^3/sec; R, m.

$$\overline{V}_x = -\frac{1}{1 + \frac{R^2}{8\eta c}\frac{\Delta P}{\Delta l}}\frac{R^2}{8\eta}\frac{\Delta P}{\Delta l}. \tag{30}$$

Formula (30) holds true only when $-\frac{R^2}{8\eta}\frac{\Delta P}{\Delta l} < c$. Otherwise, the average velocity in the pipe becomes negative. Physically this means that for large pressure gradients, the flow in the elastic pipeline loses stability and a self-oscillating regime of flow occurs [12]. Using (30), we can find the rate of flow of the fluid in the elastic pipeline:

$$Q = -\frac{1}{1 + \frac{R^2}{8\eta c}\frac{\Delta P}{\Delta l}}\frac{\pi R^4}{8\eta}\frac{\Delta P}{\Delta l} = \frac{Q_{\text{P}}}{1 + \frac{R^2}{8\eta c}\frac{\Delta P}{\Delta l}}, \tag{31}$$

where the flow rate in the rigid pipe is determined by the Poiseuille law [6] $Q_{\text{P}} = -\frac{\pi r^4}{8\eta}\frac{\Delta P}{\Delta l}$. We find the fluid velocity on the pipe axis from (28) for $r = 0$:

$$V_{\max} = -\frac{R^2}{4\eta}\left(1 + \frac{\overline{V}_x}{c}\right)\frac{\Delta P}{\Delta l}. \tag{32}$$

Solving simultaneously the system of equations (30) and (32), we can show that the relation $V_{\max} = 2\overline{V}_x$ which is true of the rigid pipeline also holds in the elastic pipeline.

Figure 3 (curve 1) plots the ratio of the average velocity of the viscous fluid to the velocity of the pressure wave c as a function of the dimensionless number $\gamma = -\frac{R^2}{8\eta c}\frac{\Delta P}{\Delta l}$ determined mainly by the pressure gradient. The plot is constructed according to formula (30). Also, the figure shows, as an example, the straight line 2 illustrating the Poiseuille law for the rigid pipe. A characteristic feature of the elastic pipeline is the nonlinear dependence of the average fluid velocity on the pressure gradient.

Figure 4 plots the rates of flow of the fluid in the pipes as functions of the radii; the plots are constructed according to formula (31) and the Poiseuille law respectively. In calculating, we have adopted the following values of the quantities: pressure gradient $-\Delta P/\Delta l = 100$ N/m^3, pressure-wave velocity $c = 8$ m/sec, and coefficient of viscosity of the fluid $\eta = 0.001$ N·sec/m^2.

NOTATION

a and *b*, integration constants; **a**, fluid-mass acceleration, m/sec^2; *c*, pressure-wave velocity, m/sec; *D*, coefficient of elasticity of the pipeline walls, N/m^2; $\mathbf{F}_{\rm fr}$, friction force, N; F_x, F_y, and F_z, projections of the friction force onto the coordinate axes, N; $\mathbf{F}_{\rm d}$, force driving the fluid, N; $\mathbf{F}_i$, forces acting on the fluid at different points *i* of the flow, N; **i**, **j**, and **k**, unit vectors in the direction of the axes *x*, *y*, and *z*; Δl, length of a portion of the pipeline, m; *m*, fluid mass, kg; *P*, pressure in the fluid, N/m^2; *Q*, rate of flow of the fluid in the pipe, m^3/sec; $Q_{\rm P}$, rate of flow of fluid in the rigid pipe, m^3/sec; *R*, inner radius of the elastic pipeline, m; *r*, radial coordinate, m; **S**, surface of the boundary of the fluid of volume *W*, m^2; $d\mathbf{S}$, differential of the contact-area vector of the fluid layers, m^2; S_x, S_y, and S_z, projections of the area vector onto the coordinate axes; in the pipe, the first projection is equal to the cross-sectional area of the pipeline, m^2; *t*, time, sec; *U*, longitudinal velocity of the external flow in the boundary layer, m/sec; **V**, fluid velocity, m/sec; V_x, V_y, and V_z, projections of the velocity vector onto the coordinate axes, m/sec; V_r, radial velocity of the fluid in the pipe, m/sec; $\overline{V}_x$, fluid velocity average over the pipeline cross section, m/sec; *W*, fluid volume, m^3; *x*, *y*, *z*, Cartesian coordinates, m; η, dynamic coefficient of viscosity, N·sec/m^2; $\nu = \eta/\rho$, kinematic viscosity of the fluid, m^2/sec; ρ, fluid density, kg/m^3; τ, tangential friction stresses, N/m^2; τ_{xz} and τ_{zx}, stress-tensor components, N/m^2. Subscripts: d, driving; fr, friction; P, Poiseuille; *xz* and *zx*, coordinate planes; max, maximum.

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